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Turing patterns created by cross-diffusion for a Holling II and Leslie-Gower type three species food chain model

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Abstract In this paper, we develop a theoretical framework for a research into spatial patterns in a three-species Holling II and Leslie-Gower type food chain model with cross-diffusion, the results of which show that the cross-diffusion induces the spatial patterns. When biological pattern formation has been concerned with the method of reaction-diffusion theory, in most of the previous works, as a precondition, the assumption of the existence of nonhomogeneous steady state is presented essentially. We give a rigorous proof to the assumption that the model has at least a nonhomogeneous stationary solution by the Leray-Schauder degree theory. Moreover, the numerical simulations for spatial pattern is also carried out, we propose a method to estimate the wavenumber of the spatial patterns.

Keywords Turing patterns · Cross-diffusion · Nonhomogeneous stationary state

1 Introduction

The formation of spatial pattern is one of the crucial areas of research in biology. The most widely studied model for spatial pattern formation is the reaction-diffusion model proposed by Alan Turing in 1952 [1]. He showed that a system of reacting and diffusing chemicals could evolve from initial near-homogeneity into a spatial pattern of chemical concentration. The phenomenon, termed diffusion-driven instability, has been shown to occur in chemistry [2–6]. Experimental results illustrate the

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formation of striped and spotted patterns, as well as more complicated patterns. Many of these patterns can be exhibited by Turing models and there is now a vast amount of theoretical and experimental literature in this area (see [7–17]).

To describe the interaction among the species in the field of population dynamics, we usually investigate the general reaction diffusion equations that has the following form

$$\frac{\partial \mathbf{u}}{\partial t} - div(D\nabla \mathbf{u}) = \mathbf{f}(\mathbf{u}),$$

where $\mathbf{u}$ is a vector $u_i(x,t),\ i = 1, 2, \ldots, m$ of species' densities, D is a $m \times m$ matrix of the diffusion coefficients, and $\mathbf{f}$ is the reaction term that indicates the interaction among the species. This paper concerns three trophic-level food chain models composed of the prey u_1, the middle predator u_2 and the top-predator u_3. The diffusion coefficients of our model are not the linear, but the following nonlinear matrix:

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & \frac{d_3 \partial(u_3(1+d_4u_2))}{\partial u_2} & \frac{d_3 \partial(u_3(1+d_4u_2))}{\partial u_3} \end{pmatrix}.$$

And the reaction term $\mathbf{f}$ has the functional response of Hollowing II and Leslie-Gower scheme, which is first given out in [18]. The mathematical model is as follows:

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1 \left(1 - u_1 - \frac{u_2}{u_1+l}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = u_2 \left(-b + \frac{au_1}{u_1+l} - eu_2 - \frac{u_3}{u_2+m}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u_3}{\partial t} - d_3 \Delta(u_3 + d_4u_2u_3) = u_3 \left(c - \frac{u_3}{u_2+n}\right), & x \in \Omega, \quad t > 0, \\ \partial_n u_1 = \partial_n u_2 = \partial_n u_3 = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{1.1}$$

Our model has the following biological meanings:

- Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. ∂_n is the directional derivative normal to $\partial\Omega$. The homogeneous Neumann boundary condition indicates that there are zero population flux across the boundary.
- b and c respectively denote the intrinsic growth rates.
- e represents the internal competition coefficients of species u_2.
- a indicates the maximum value which unity reduction rate of species u_2 can attain.
- The functional response $\frac{1}{u_1+l}$ and $\frac{1}{u_2+m}$ belong to Hollowing II type.
- The functional response $\frac{1}{u_2+n}$ belongs to Leslie-Gower scheme.
- The nonlinear diffusion term means that the disperse direction of top-predator not only contains the self-diffusion, in which way the species move from high density domain to low density, but also contains cross-diffusion. Especially, the top-predator u_3 diffuses with flux

$$\mathbf{J} = -\nabla(d_3u_3 + d_3d_4u_2u_3) = -d_3d_4u_3\nabla u_2 - (d_3 + d_3d_4u_2)\nabla u_3.$$

We observe that, as $-d_3d_4u_3 < 0$, the part $-d_3d_3u_3\nabla u_2$ of the flux is directed toward the decreasing population density of the prey u_2. When the predator is chasing the prey, the flux should be directed toward the increasing population density of the prey as in Kareiva and Odell [19]. However, in certain kinds of predator-prey relationships, a great number of prey species form a huge group to protect themselves from the attack of predator, see also [20,21] and references therein for more detail.

The general food chains have largely been studied, see Lin [22–24] and the refernces therein. The model (1.1) is first proposed by Aziz-Alaoui et al. [25]. In recent years, they have studied the modified models in [26] and the refernces therein. However, they have not studied the issue of Turing patterns. This paper aims to study the Turing patterns.

In this paper, using the method like in [27], we shall give a rigorous proof to the assumption. Many recent papers have dealt specifically with cross-diffusion as means of creating spatial patterns, as for example the work of [28–31]. The rest of this paper is organized as follows. In Sect. 2, we demonstrate that cross-diffusion destabilizes a uniform equilibrium which is stable for the kinetic and self-diffusion reaction systems. The configuration of the spatial pattern for (1.1) is given by virtue of numerical method proposed in [32–34]. In Sect. 3, we show that (1.1) has nonhomogeneous steady state by using the Leray-Schauder degree theory. Section 4 is devoted to some conclusions.

2 Spatial patterns formation

In this section, our main objective is to look for the conditions on the parameter values such that the positive stationary uniform solution is linearly stable in the absence of the cross-diffusion but unstable in the presence of the cross-diffusion. This case is the well-known phenomenon of cross-diffusion driven instability. Moreover, we use numerical methods to investigate the spatial patterns.

2.1 Linear analysis of instability

In [26], it has been shown that assume that

$$(H_1) \quad e\left(1-\frac{1}{l}\right) < \frac{a}{l^2},$$

system (1.1) has the unique positive stationary uniform solution, we denote it by $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$. Throughout this paper, we assume that (H_1) holds. First, we give the conditions such that the following system (2.1), the corresponding kinetic equations of (1.1), is locally stable.

$$\begin{cases} \frac{du_1}{dt} = u_1\left(1-u_1-\frac{u_2}{u_1+l}\right), \\ \frac{du_2}{dt} = u_2\left(-b+\frac{au_1}{u_1+l}-eu_2-\frac{u_3}{u_2+m}\right), \\ \frac{du_3}{dt} = u_3\left(c-\frac{u_3}{u_2+n}\right). \end{cases} \tag{2.1}$$

Theorem 2.1 *The stationary uniform solution* $\tilde{\mathbf{u}}$ *of* (2.1) *is locally asymptotically stable, if the parameters of* (1.1) *satisfy*

$$(H_2) \qquad \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2} < 1 \quad \textit{and} \quad \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2} < e.$$

Proof It is sufficient to show that the characteristic value of (2.1) is negative. For simplicity, throughout this paper, we denote

$$\mathbf{G}(\mathbf{u}) = \begin{pmatrix} G_1(\mathbf{u}) \\ G_2(\mathbf{u}) \\ G_3(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} u_1 g_1(\mathbf{u}) \triangleq u_1 = u_1(1-u_1-\frac{u_2}{u_1+l}) \\ u_2 g_2(\mathbf{u}) \triangleq u_2(-b+\frac{au_1}{u_1+l}-\frac{u_3}{u_2+m}) \\ u_3 g_3(\mathbf{u}) \triangleq u_3(c-\frac{u_3}{u_2+n}) \end{pmatrix}. \tag{2.2}$$

Noticing $\mathbf{G}(\tilde{\mathbf{u}}) = 0$, a direct calculation yields

$$\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} \tilde{u}_1(-1+\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}) & -\frac{\tilde{u}_1}{\tilde{u}_1+l} & 0 \\ \frac{al\tilde{u}_2}{(\tilde{u}_1+l)^2} & \frac{\tilde{u}_2\tilde{u}_3}{(\tilde{u}_2+m)^2} & -\frac{\tilde{u}_2}{\tilde{u}_2+m} \\ 0 & \frac{\tilde{u}_3^2}{(\tilde{u}_2+n)^2} & -\frac{\tilde{u}_3}{\tilde{u}_2+n} \end{pmatrix}. \tag{2.3}$$

The characteristic polynomial of $\mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})$ is

$$\varphi(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,$$

where

$$\begin{aligned} A_1 &= \tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right) + \frac{\tilde{u}_2\tilde{u}_3}{(\tilde{u}_2+m)^2} + \frac{\tilde{u}_3}{\tilde{u}_2+n}, \\ A_2 &= \tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)\left(-\frac{\tilde{u}_2\tilde{u}_3}{(\tilde{u}_2+m)^2} + \frac{\tilde{u}_3}{\tilde{u}_2+n}\right) \\ &\quad -\frac{\tilde{u}_2\tilde{u}_3}{(\tilde{u}_2+m)^2}\frac{\tilde{u}_3}{\tilde{u}_2+n} + \frac{\tilde{u}_1}{\tilde{u}_1+l}\frac{al\tilde{u}_2}{(\tilde{u}_1+l)^2} + \frac{\tilde{u}_2}{\tilde{u}_2+m}\frac{\tilde{u}_3^2}{(\tilde{u}_2+n)^2} \\ A_3 &= \tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)\left[-\frac{\tilde{u}_2\tilde{u}_3}{(\tilde{u}_2+m)^2}\frac{\tilde{u}_3}{\tilde{u}_2+n} + \frac{\tilde{u}_2\tilde{u}_3^2}{(\tilde{u}_2+n)^2(\tilde{u}_2+m)^2}\right] \\ &\quad +\frac{al\tilde{u}_1\tilde{u}_2\tilde{u}_3}{(\tilde{u}_1+l)^3(\tilde{u}_2+n)}. \end{aligned}$$

By (H_2), it is easy to verify that A_1, A_2, A_3 are positive, and $A_1A_2 - A_3 > 0$. Thus, it follows from the Routh-Hurwitz criterion that $\tilde{\mathbf{u}}$ is locally asymptotically stable. □

We now consider system (1.1) without cross-diffusion in the following form

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1 \left(1 - u_1 - \frac{u_2}{u_1+l}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = u_2 \left(-b + \frac{a u_1}{u_1+l} - e u_2 - \frac{u_3}{u_2+m}\right), & x \in \Omega, \quad t > 0, \\ \frac{\partial u_3}{\partial t} - d_3 \Delta u_3 = u_3 \left(c - \frac{u_3}{u_2+n}\right), & x \in \Omega, \quad t > 0, \\ \partial_n u_1 = \partial_n u_2 = \partial_n u_3 = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{2.4}$$

In order to discuss the locally asymptotically stability of the parabolic equation (2.4), we set up the following notation, similarly as in [27,28,35].

Notation 2.1 *Let* $0 = \mu_1 < \mu_2 < \cdots \to \infty$ *be the eigenvalues of* $-\Delta$ *on* Ω *under no-flux boundary condition, and* $E(\mu_i)$ *be the space of eigenfunctions corresponding to* μ_i. *We define the following space decomposition*

(i) $\mathbf{X_{ij}} := \{\mathbf{c} \cdot \phi_{ij} \ : \ \mathbf{c} \in \mathbb{R}^3\}$, *where* $\{\phi_{ij}\}$ *are orthonormal basis of* $E(\mu_i)$ *for* $j = 1, \ldots, dim E(\mu_i)$.
(ii) $\mathbf{X} := \{\mathbf{u} \in [C^1(\bar{\Omega})]^3 \ : \ \partial_n u_1 = \partial_n u_2 = \partial_n u_3 = 0 \ \ on \ \partial\Omega\}$, *and so* $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X_i}$, *where* $\mathbf{X_i} = \bigoplus_{j=1}^{dim E(\mu_i)} \mathbf{X_{ij}}$.

Theorem 2.2 *The stationary uniform solution* $\tilde{\mathbf{u}}$ *of* (2.4) *is locally asymptotically stable if* (H_2) *holds.*

Proof The linearization of (2.4) at $\tilde{\mathbf{u}}$ can be expressed by

$$\mathbf{u_t} = (D\Delta + \mathbf{G_u}(\tilde{\mathbf{u}}))\mathbf{u},$$

where

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}. \tag{2.5}$$

According to Notation 2.1, $\mathbf{X_i}$ is invariant under the operator $D\Delta + \mathbf{G_u}(\tilde{\mathbf{u}})$, and λ is an eigenvalue of this operator on $\mathbf{X_i}$, if and only if it is an eigenvalue of the matrix $-\mu_i D + \mathbf{G_u}(\tilde{\mathbf{u}})$.

After some calculation the characteristic polynomial of $-\mu_i D + \mathbf{G_u}(\tilde{\mathbf{u}})$ is given by

$$\psi_i(\lambda) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3$$

where

$$
\begin{aligned}
B_1 &= \tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)+\tilde{u}_2\left(e-\frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right)+\frac{\tilde{u}_3}{\tilde{u}_2+n}+\mu_i(d_1+d_2+d_3),\\
B_2 &= \left(\tilde{u}_2\left(e-\frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right)+d_2\mu_i\right)\left(\frac{\tilde{u}_3}{\tilde{u}_2+n}+d_3\mu_i\right)+\frac{\tilde{u}_1}{\tilde{u}_1+l}\frac{al\tilde{u}_2}{(\tilde{u}_1+l)^2}\\
&\quad+\left(\tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)+d_1\mu_i\right)\\
&\quad\times\left(d_2\mu_i+d_3\mu_i+\tilde{u}_2\left(e-\frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right)+\frac{\tilde{u}_3}{\tilde{u}_2+n}\right)+\frac{\tilde{u}_2}{\tilde{u}_2+m}\frac{\tilde{u}_3^2}{(\tilde{u}_2+n)^2},\\
B_3 &= \left(\tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)+d_1\mu_i\right)\left(\tilde{u}_2\left(e-\frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right)+d_2\mu_i\right)\\
&\quad\times\left(\frac{\tilde{u}_3}{\tilde{u}_2+n}+d_3\mu_i\right)+\frac{al\tilde{u}_1\tilde{u}_2}{(\tilde{u}_1+l)^3}\left(\frac{\tilde{u}_3}{\tilde{u}_2+n}+d_3\mu_i\right)\\
&\quad+\frac{\tilde{u}_2\tilde{u}_3^2\left(\tilde{u}_1\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)+d_1\mu_i\right)}{(\tilde{u}_2+m)(\tilde{u}_2+n)^2}.
\end{aligned}
$$

By using (H_2), it is easy to verify that B_1, B_2, B_3 are positive, $B_1B_2 - B_3 > 0$. It thus follows from the Routh-Hurwitz criterion that, for each $i \geq 1$, the three roots $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$ of $\psi_i(\lambda) = 0$ all have negative real parts. Thus the proof is finished by Theorem 5.1.1 of [36]. □

From Theorem 2.2, we see that only adding the self-crossing to the ODE system (2.1), the positive stationary uniform solution is also locally stable, which means that Turing instability has not happened. Considering the effect of the cross-diffusion, we get the following theorem.

Theorem 2.3 *Assume that* $d_4 > 0$,

$$
(H_3)\quad \frac{d_4\tilde{u}_3}{\tilde{u}_2+m}\left(1-\frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right) > \left(e+\frac{al}{(\tilde{u}_1+l)^3}\right)(1+d_4\tilde{u}_2)
$$

and (H_2) *hold, if* $\mu_2 < \tilde{\mu}$, *where* μ_2 *is given in Notation 2.1,* $\tilde{\mu}$ *will be given in* (2.8). *Then there exists a positive constant* d_2^*, *when* $d_3 \geq d_3^*$, *the stationary uniform solution* $\tilde{\mathbf{u}}$ *of* (1.1) *is unstable.*

Proof For simplicity, we denote that $\Phi(\mathbf{u}) = (d_1u_1, d_2u_2, d_3u_3(1+d_4u_2))^T$. Linearizing system (1.1) at $\tilde{\mathbf{u}}$ and then we have

$$
\mathbf{u_t} = (\Phi_u\Delta + \mathbf{G_u}(\tilde{\mathbf{u}}))\mathbf{u},
$$

where

$$
\Phi_u = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & d_3d_4\tilde{u}_3 & d_3+d_3d_4\tilde{u}_2 \end{pmatrix}.
$$

By some calculation, the characteristic polynomial of $-\mu_i\Phi_u + \mathbf{G_u}(\tilde{\mathbf{u}})$ is written by

$$\psi_i(\lambda) = \lambda^3 + \bar{B}_1\lambda^2 + \bar{B}_2\lambda + \bar{B}_3,$$

where

$$\begin{aligned}
\bar{B}_1 &= \tilde{u}_1\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right) + \tilde{u}_2\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right) + \frac{\tilde{u}_3}{\tilde{u}_2+n} + \mu_i(d_1+d_2+d_3),\\
\bar{B}_2 &= \left(\tilde{u}_2(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}) + d_2\mu_i\right)\left(\frac{\tilde{u}_3}{\tilde{u}_2+n} + d_3\mu_i\right) + \frac{\tilde{u}_1}{\tilde{u}_1+l}\frac{al\tilde{u}_2}{(\tilde{u}_1+l)^2}\\
&\quad + \left(\tilde{u}_1\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right) + d_1\mu_i\right)\left(d_2\mu_i + d_3\mu_i + \tilde{u}_2\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right)\right.\\
&\quad \left. + \frac{\tilde{u}_3}{\tilde{u}_2+n}\right) + \frac{\tilde{u}_2}{\tilde{u}_2+m}\left(\frac{\tilde{u}_3^2}{(\tilde{u}_2+n)^2} + d_3d_4\tilde{u}_3\mu_i\right),\\
\bar{B}_3 &= \left(\tilde{u}_1(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}) + d_1\mu_i\right)\left(\tilde{u}_2\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right) + d_2\mu_i\right)\\
&\quad \times\left(\frac{\tilde{u}_3}{\tilde{u}_2+n} + d_3\mu_i\right) + \frac{\tilde{u}_2}{\tilde{u}_2+m}\left(\tilde{u}_1\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right) + d_1\mu_i\right)\\
&\quad \times\left(\frac{\tilde{u}_3^2}{(\tilde{u}_2+n)^2} + d_3d_4\tilde{u}_3\mu_i\right) + \frac{al\tilde{u}_1\tilde{u}_2}{(\tilde{u}_1+l)^3}\left(\frac{\tilde{u}_3}{\tilde{u}_2+n} + (d_3 + d_3d_4\tilde{u}_2)\mu_i\right).
\end{aligned}$$

Let $\lambda_1(\mu_i), \lambda_2(\mu_i), \lambda_3(\mu_i)$ be the three roots of $\psi_i(\lambda) = 0$, then $\lambda_1(\mu_i)\lambda_2(\mu_i)\lambda_3(\mu_i) = -\bar{B}_3$. In order to have at least one $Re\lambda_i(\mu_i) > 0$, it is sufficient that $\bar{B}_3 < 0$.

In the following we shall find out the conditions such that $\bar{B}_3 < 0$. In fact $\bar{B}_3 = det(\mu_i\Phi_u - \mathbf{G_u}(\tilde{\mathbf{u}}))$, and after some calculation, we have

$$\bar{B}_3 = Q_3\mu_i^3 + Q_2\mu_i^2 + Q_1\mu_i - det(\mathbf{G_u}), \tag{2.6}$$

where

$$\begin{aligned}
Q_3 &= d_1d_2d_3(1 + d_4\tilde{u}_2),\\
Q_2 &= -d_1d_3d_4\tilde{u}_3\frac{\tilde{u}_2}{\tilde{u}_2+m} + d_2d_3(1+d_4\tilde{u}_2)\tilde{u}_1\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)\\
&\quad + d_1d_3(1+d_4\tilde{u}_2)\tilde{u}_2\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right) + d_1d_2\frac{\tilde{u}_2}{\tilde{u}_2+n},\\
Q_1 &= d_1\tilde{u}_2\frac{\tilde{u}_3}{\tilde{u}_2+n}\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right) + d_2\frac{\tilde{u}_1\tilde{u}_3}{\tilde{u}_2+n}\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)\\
&\quad + d_3(1+d_4\tilde{u}_2)\tilde{u}_1\tilde{u}_2\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right)\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2+m)^2}\right)\\
&\quad - d_3d_4\tilde{u}_3\frac{\tilde{u}_1\tilde{u}_3}{\tilde{u}_2+n}\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1+l)^2}\right) + d_1\frac{\tilde{u}_2\tilde{u}_3^2}{(\tilde{u}_2+m)(\tilde{u}_2+n)^2}\\
&\quad + d_3(1+d_4\tilde{u}_2)\frac{al\tilde{u}_1\tilde{u}_2}{(\tilde{u}_1+l)^2}
\end{aligned}$$

$$
\begin{aligned}
-det(\mathbf{G_u}) = \tilde{u}_1 &\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1 + l)^2}\right) \\
&\left[\tilde{u}_2 \left(e - \frac{\tilde{u}_3}{(\tilde{u}_2 + m)^2}\right) \frac{\tilde{u}_3}{\tilde{u}_2 + n} + \frac{\tilde{u}_2 \tilde{u}_3^2}{(\tilde{u}_2 + n)^2 (\tilde{u}_2 + m)^2}\right] \\
&+ \frac{al\tilde{u}_1 \tilde{u}_2 \tilde{u}_3}{(\tilde{u}_1 + l)^3 (\tilde{u}_2 + n)}.
\end{aligned}
$$

Let $\tilde{Q}(\mu) = Q_3\mu^3 + Q_2\mu^2 + Q_1\mu - det(\mathbf{G_u})$ and let $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$ be the three roots of $\tilde{Q}(\mu) = 0$ with $Re(\tilde{\mu}_1) \le Re(\tilde{\mu}_2) \le Re(\tilde{\mu}_3)$. Then $\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3 = det(\mathbf{G_u}) < 0$. Notice that $Q_3 > 0$. Thus, one of $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{\mu}_3$ is real and negative, and the product of the other two is positive.

Consider the following limits

$$
\begin{aligned}
\lim_{d_3 \to \infty} \frac{Q_3}{d_3} &= d_1 d_2 (1 + d_4 \tilde{u}_2) \triangleq b_3, \\
\lim_{d_3 \to \infty} \frac{Q_2}{d_3} &= d_2 (1 + d_4 \tilde{u}_2) \tilde{u}_1 \left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1 + l)^2}\right) \\
&\quad + d_1 (1 + d_4 \tilde{u}_2) \tilde{u}_2 \left(e - \frac{\tilde{u}_3}{(\tilde{u}_2 + m)^2}\right) \triangleq b_2, \\
\lim_{d_3 \to \infty} \frac{Q_1}{d_3} &= \left(\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2 + m)^2}\right)\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1 + l)^2}\right) + \frac{al}{(\tilde{u}_1 + l)^3}\right)(1 + d_4 \tilde{u}_2) \\
&\quad - \frac{d_4 \tilde{u}_3}{\tilde{u}_2 + m} \left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1 + l)^2}\right) \triangleq b_1.
\end{aligned}
\tag{2.7}
$$

Note that

$$
\begin{aligned}
\lim_{d_3 \to \infty} \frac{\tilde{Q}(\mu)}{d_3} &= b_3\mu^3 + b_2\mu^2 + b_1\mu \\
&= \mu(b_3\mu^2 + b_2\mu + b_1).
\end{aligned}
$$

From the conditions (H_1) and (H_3), we can have

$$
\begin{aligned}
&\left(\left(e - \frac{\tilde{u}_3}{(\tilde{u}_2 + m)^2}\right)\left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1 + l)^2}\right) + \frac{al}{(\tilde{u}_1 + l)^3}\right)(1 + d_4 \tilde{u}_2) \\
&< \frac{d_4 \tilde{u}_3}{\tilde{u}_2 + m} \left(1 - \frac{\tilde{u}_2}{(\tilde{u}_1 + l)^2}\right),
\end{aligned}
$$

it follows that $b_1 < 0 < b_3$, which means that the equation $\lim_{d_3 \to \infty} \frac{\tilde{Q}(\mu)}{d_3} = 0$ have one positive root, one negative root and zero. A continuity argument shows that, when d_3 is enough large, $\tilde{\mu}_1$ is real and negative. Furthermore, as $\tilde{\mu}_2\tilde{\mu}_3 > 0$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$ are real and positive, and

$$\begin{aligned} &\lim_{d_2\to\infty}\tilde{\mu}_1=\frac{-b_2-\sqrt{b_2^2-4b_1b_3}}{2b_3}<0, \qquad \lim_{d_2\to\infty}\tilde{\mu}_2=0,\\ &\lim_{d_2\to\infty}\tilde{\mu}_3=\frac{-b_2+\sqrt{b_2^2-4b_1b_3}}{2b_3}\triangleq\tilde{\mu}>0. \end{aligned} \tag{2.8}$$

Hence there exists a positive number d_3^* such that, when $d_3 \geq d_3^*$, the following holds

$$\tilde{Q}(\mu)<0 \ \texttt{ when } \ \mu\in(-\infty,\tilde{\mu}_1)\cup(\tilde{\mu}_2,\tilde{\mu}_3). \tag{2.9}$$

Since $0<\mu_2<\tilde{\mu}$, then $\mu_2\in(\tilde{\mu}_2,\tilde{\mu}_3)$, it follows that $\tilde{Q}(\mu_2)<0$. Thus we show that $\bar{B}_3<0$, the proof is completed. □

Remark 2.1 When $m=+\infty$, the model (1.1) reduces to a predator-prey model and a single species Logistic model. In view of (H_2), the condition (H_3) can not hold. Thus we can not use Theorem 2.3 to obtain Turing instability. The existence of top-predator is essential for the cross-diffusion induced pattern formation.

From the above theorems we see that the cross-diffusion destabilize the stationary uniform solution, so the spatial patterns for the Holling II and Leslie-Gower type food chain model generate.

2.2 Numerical calculations

In the above subsection, the condition (H_3) can induce the stationary uniform solution Turing instability, the so-called Turing parameter space. In this subsection, using numerical methods, we illustrate that the cross-diffusion induce spatial patterns. Moreover, because our model is a three-component reaction diffusion system, we can not compute the explicit formulas about the wavenumber of spatial patterns. But as a special numerically example is concerned, we can compute the wavenumber of spatial patterns. We also argue the dispersion relation and compare the patterns for different domain scale.

Throughout this subsection we assume the region of system (1.1) is a rectangular domain $\Omega=[0,L]\times[0,L]\subset\mathbb{R}^2$. By the definition of the eigenvalue μ_i in Notation 2.1, $\mu_i=(\frac{(i-1)\pi}{L})^2$, $i\in\mathbb{N}$, hence $\mu_2=(\frac{\pi}{L})^2$. Due to the complicated algebraic form of $\tilde{\mu}$ in (2.8), we need to choose the proper parameters such that the spatial pattern generates, which is induced by the cross-diffusion term $d_3d_4\Delta(u_2u_3)$. Thus, in the system (1.1) we take

$$e=1,\ l=2,\ b=1,\ c=8,\ a=50, \tag{2.10}$$

$$d_1=0.1,\ d_2=0.1,\ d_3=4.85,\ d_4=3, \tag{2.11}$$

these special parameters belong to the Turing space.

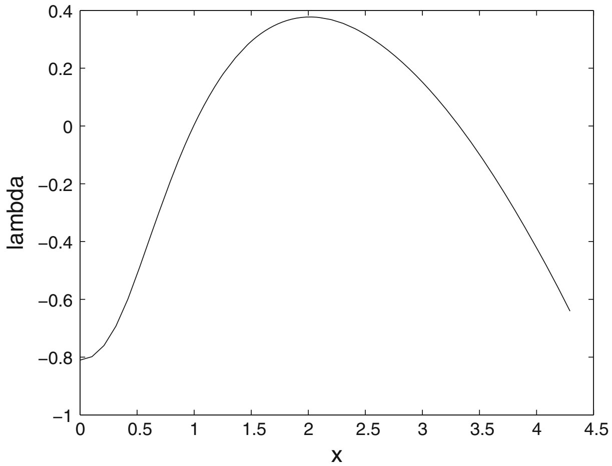

Fig. 1 Eigenvalues of the linearized characteristic matrix of Eq. (1.1). In this case $L = 30$, then $\mu_i = (\frac{(i-1)\pi}{L})^2$, where $k = \frac{(i-1)\pi}{L}$, $i = 1, 2, \ldots$ are the wavenumber. When the real part of the eigenvalue λ reaches the positive maximum point, the critical value of k is $k^* = 1.9897$, which is the approximate value of the wavenumber of the pattern

In Fig. 1 we show that the real part of the eigenvalues λ as a function of wavenumber k. From the data of Fig. 1, when $k^* = 1.9897$, the eigenvalue of the temporal pattern reaches its maximal value. Thus the approximate value of the wavenumber is 1.9897. Note that there can be a number of different type spatial pattern under the same number. Notice the 3 dimensional reaction diffusion system we discuss, we do not know which parameter dominate in pattern selection. Applying the modified numerical method in [32], we consider system (1.1) in a fixed domain and solve it on a grid with 100×100 sites by a simple Euler method with a time step of $\Delta t = 0.001$, and by discretizing the Laplacian in the grid with lattice sites denoted by (i, j). The form is

$$\nabla^2 w|_{(i,j)} = \frac{1}{s^2}(a_l(i, j)w(i-1, j) + a_r(i, j)w(i+1, j) \\ + a_d(i, j)w(i, j-1) + a_u(i, j)w(i, j+1) - 4w(i, j)), \tag{2.12}$$

where s is the lattice constant and the matrix elements of a_l, a_r, a_d, a_u are unity except at the boundary. When (i, j) is at the left boundary, that is $i = 0$, we define $a_l(i, j)w(i-1, j) \equiv w(i+1, j)$, which guarantees zero-flux of reactants in the left boundary. Similarly we define $a_r(i, j)$, $a_d(i, j)$, $a_u(i, j)$ such that the boundary is no-flux. Notice that w of (2.12) can be expressed by u_1, u_2 or the function $u_3 + d_4 u_2 u_3$.

In Figs. 2 and 3, we respectively give out the different spatial patterns according to the scale of domain $L = 30$ and $L = 60$ under the parameters are the same in (2.10) and (2.11). We notice that the all spatial patterns are stripe.

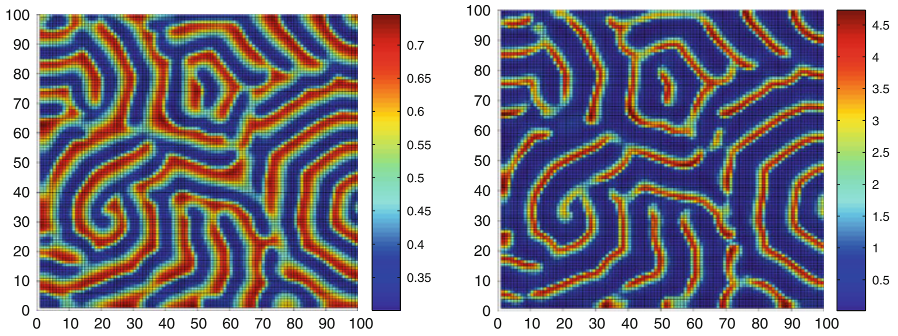

Fig. 2 Spatial patterns obtained with (1.1) species u_1 when the parameters are in (2.10) and (2.11) and the scale $L = 30$. The *left figure* and the *right figure* are respectively illustrated species u_1 and u_2. The steps of time iteration is 200,000

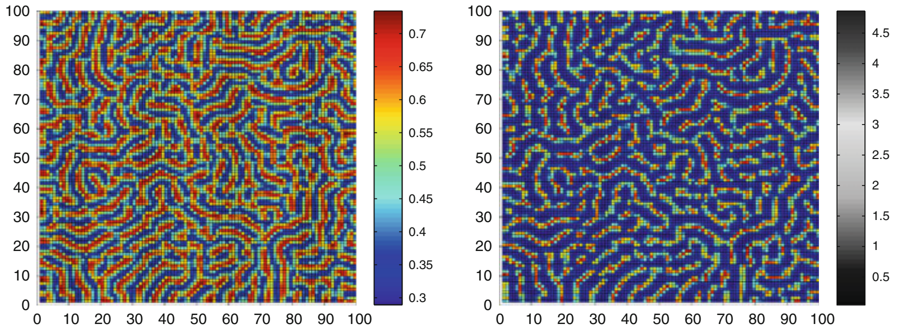

Fig. 3 Spatial patterns obtained with (1.1) species u_1 when the parameters are in (2.10) and (2.11) and the scale $L = 60$. The *left figure* and the *right figure* are respectively illustrated species u_1 and u_2. The steps of time iteration is 200,000

3 Nonhomogeneous steady states

In the above section, by means of numerical simulations, we show that the stationary uniform solution is unstable when the cross-diffusion appears. In this section, to discern the reason why spatial pattern happens, we shall show that (1.1) admits the nonhomogeneous steady state.

First we consider the corresponding steady state problem of (1.1) in the form

$$\begin{cases} -d_1\Delta u_1 = G_1(\mathbf{u}), & x \in \Omega, \\ -d_2\Delta u_2 = G_2(\mathbf{u}), & x \in \Omega, \\ -d_3\Delta(u_3 + d_4 u_2 u_3) = G_3(\mathbf{u}), & x \in \Omega, \\ \partial_n u_1 = \partial_n u_2 = \partial_n u_3 = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

In the following text, the generic constants C^*, $\underline{C}$, $\overline{C}$, will depend on the domain Ω. However, as Ω is fixed, we shall not mention this dependence explicitly. Also, for convenience, we denote the parameters a, b, c, e, l, m, n collectively by Λ.

3.1 A priori estimates

The aim of this part is to give a priori upper and lower bounds for the positive solutions to (3.1). To this purpose, we cite two known results.

Proposition 3.1 (Harnack Inequality [37]) *Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\bar{\Omega})$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant C_* which depends only on $\|c\|_\infty$ such that*

$$\max_{\Omega} w \le C_* \min_{\Omega} w$$

Proposition 3.2 (Maximum Principle [38]) *Let $g \in C(\Omega \times \mathbb{R}^1)$ and $b_{ij} \in C(\bar{\Omega})$, $j = 1, 2, \ldots, N$.*

(i) *If $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{cases} \Delta w(x) + \sum_{j=1}^{N} b_j(x) w_{x_j} + g(x, w(x)) \ge 0, & x \in \Omega, \\ \partial_n w(x) \le 0, & x \in \partial\Omega, \end{cases}$$

and $w(x_0) = \max_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \ge 0$.

(ii) *If $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{cases} \Delta w(x) + \sum_{j=1}^{N} b_j(x) w_{x_j} + g(x, w(x)) \le 0, & x \in \Omega, \\ \partial_n w(x) \ge 0, & x \in \partial\Omega, \end{cases}$$

and $w(x_0) = \min_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \le 0$.

Theorem 3.1 (Upper bound) *Let D_1, D_2, D_3 and D_4 be fixed positive constants. Then there exist positive constants $C^*(\Lambda, D_1, D_2, D_3, D_4)$ and $\overline{C}(\Lambda, D_1, D_2, D_3, D_4)$ such that, when $d_1 \ge D_1$, $d_2 \ge D_2$, $d_3 \ge D_3$ and $d_4 \le D_4$, the positive solution $\mathbf{u} = (u_1, u_2, u_3)^T$ of (3.1) satisfies*

$$\begin{aligned} &\max_{\bar{\Omega}} u_i \le \overline{C}(\Lambda, D_1, D_2, D_3, D_4) \\ &\max_{\bar{\Omega}} u_i \le C^*(\Lambda, D_1, D_2, D_3, D_4) \min_{\bar{\Omega}} u_i, \ i = 1, 2, 3. \end{aligned} \tag{3.2}$$

Proof First, a direct application of the maximum principle to the first equation of (3.1) gives $\max_{\bar{\Omega}} u_1 \le 1$, then to the second equation of (3.1) gives $max_{\bar{\Omega}} u_2 \le \frac{a}{e(1+l)}$.

Define $\varphi(x) = d_3(d_4 u_2 u_3 + u_3)$. Let $x_1 \in \bar{\Omega}$ be such that $\varphi(x_1) = \max_{\bar{\Omega}} \varphi$, applying the maximum principle to the third equation of (3.1) yields $u_3(x_1)(c - \frac{u_3(x_1)}{u_2(x_1)+n} \ge 0$, which implies

$$u_3(x_1) \le c(u_2(x_1) + n) \le c\left(\frac{a}{e(1+l)} + n\right). \tag{3.3}$$

From the definition of $\varphi(x)$ and in view of $d_4 \leq D_4$ we have

$$\begin{aligned}\max_{\bar{\Omega}} u_3 &\leq \frac{1}{d_3}\max_{\bar{\Omega}}\varphi = \frac{1}{d_3}\varphi(x_1) = u_3(x_1) + d_4 u_2(x_1)u_3(x_1)\\ &\leq u_3(x_1) + d_4 \max_{\bar{\Omega}} u_2 u_3(x_1) \leq c\left(\frac{a}{e(1+l)} + n\right)\left(1 + D_4\frac{a}{e(1+l)}\right).\end{aligned} \tag{3.4}$$

Let $\overline{C}(\Lambda, d) = \max\{1, \frac{a}{e(1+l)}, c(\frac{a}{e(1+l)} + n)(1 + D_4\frac{a}{e(1+l)})\}$, then we have

$$\max_{\bar{\Omega}} u_i \leq \overline{C}(\Lambda, D_1, D_2, D_3, D_4).$$

Next we shall show $\max_{\bar{\Omega}} u_i \leq C^*(\Lambda, D_1, D_2, D_3, D_4)\min_{\bar{\Omega}} u_i$. Since u_1, u_2 and u_3 are bounded, directly applying the Harnack Inequality to the first and the second equation of (3.1), we obtain

$$\max_{\bar{\Omega}} u_i \leq C_i^*(\Lambda, D_1, D_2, D_3, D_4)\min_{\bar{\Omega}} u_i, \qquad i = 1, 2.$$

Similarly, define $\varphi(x) = d_3(d_4u_2u_3 + u_3)$ and we have

$$\begin{cases}\Delta\varphi + c(x)\varphi = 0, & x \in \Omega,\\ \partial_n\varphi = 0, & x \in \partial\Omega.\end{cases}$$

where $c(x) = \frac{cu_2 - u_3 - cn}{d_3(1+d_4u_2)(u_2+n)}$ is bounded, the Harnack Inequality gives $\max_{\bar{\Omega}}\varphi \leq C_3^*\min_{\bar{\Omega}}\varphi$ for some positive constant $C_3^* = C_3^*(\Lambda, D_1, D_2, D_3, D_4)$, and

$$\frac{\max_{\bar{\Omega}} u_3}{\min_{\bar{\Omega}} u_3} \leq \frac{\max_{\bar{\Omega}}\varphi\max_{\bar{\Omega}}(1 + d_4u_2)}{\min_{\bar{\Omega}}\varphi\min_{\bar{\Omega}}(1 + d_4u_2)} \leq C_3^*\frac{\max_{\bar{\Omega}} u_2}{\min_{\bar{\Omega}} u_2} \leq C_2^*C_3^*.$$

Let $C^*(\Lambda, D_1, D_2, D_3, D_4) = \max\{C_1^*, C_2^*, C_2^*C_3^*\}$, then the proof is completed. □

Before showing the lower bound, we give the following preliminary result.

Lemma 3.1 *Let $d_{i,m}$, $i = 1, 2, 3, 4$, be positive constants, $m = 1, 2, \ldots$, and $\mathbf{u_m} = (u_{1m}, u_{2m}, u_{3m})^T$ be the corresponding positive solution of (3.1) with $d_i = d_{i,m}$. If $\mathbf{u_m} \to \overline{\mathbf{u}}$ as $m \to \infty$ and $\overline{\mathbf{u}}$ is a constant vector, then $\overline{\mathbf{u}} = \tilde{\mathbf{u}}$. Recall that $\tilde{\mathbf{u}}$ is the unique positive solution of $\mathbf{g}(\mathbf{u}) = 0$.*

Proof It is easy to see that for all m, $\int_\Omega u_{1m}g_1(\mathbf{u_m})dx = 0$. If $g_1(\overline{\mathbf{u}}) > 0$, then $g_1(\mathbf{u_m}) > 0$ when m is large. But since u_{1m} is positive, this is impossible. Similarly, $g_1(\overline{\mathbf{u}}) < 0$ is impossible. Therefore, $g_1(\overline{\mathbf{u}}) = 0$. The same argument shows that $g_2(\overline{\mathbf{u}}) = g_3(\overline{\mathbf{u}}) = 0$. Consequently, $\overline{\mathbf{u}} = \tilde{\mathbf{u}}$. □

Theorem 3.2 (Lower bound) *Let D_1, D_2, D_3 and D_4 be fixed positive constants. Then there exist positive constants $\underline{C}(\Lambda, D_1, D_2, D_3, D_4)$ such that, when $d_1 \geq D_1$,*

$d_2 \geq D_2$, $d_3 \geq D_3$ and $d_4 \leq D_4$, the positive solution $\mathbf{u} = (u_1, u_2, u_3)^T$ of (3.1) satisfies

$$\min_{\bar{\Omega}} u_i \geq \underline{C}(\Lambda, D_1, D_2, D_3, D_4), \quad i = 1, 2, 3. \tag{3.5}$$

Proof First, integrating the second equation of (3.1) in Ω and taking account of the nonhomogeneous Neumann boundary condition, we have

$$\int_{\Omega} u_2 \left(-b + \frac{au_1}{u_1 + l} - eu_2 - \frac{u_3}{u_2 + m} \right) = 0.$$

Thus, there exists $x_0 \in \Omega$ such that $\frac{au_1(x_0)}{u_1(x_0)+l} = b + eu_2(x_0) + \frac{u(x_0)}{u_2(x_0)+m}$, which implies that $u_1(x_0) \geq \frac{bl}{a}$. By use of the Harnack equality, we have

$$\min_{\bar{\Omega}} u_1(x) \geq \frac{1}{C^*} \frac{bl}{a}.$$

Similarly, integrating the third equation of (3.1) in Ω, we have

$$\min_{\bar{\Omega}} u_3(x) \geq \frac{1}{C^*} cn.$$

In view of (3.2), in order to establish Theorem 3.2, it suffices to show that

$$\max_{\bar{\Omega}} u_2 \geq \underline{C}_1(\Lambda, D_1, D_2, D_3, D_4). \tag{3.6}$$

Suppose that (3.6) fails. Then there exist sequences $\{d_{1,m}, d_{2,m}, d_{3,m}, d_{4,m}\}_{m=1}^{\infty}$ with $d_{1,m} \geq D_1, d_{2,m} \geq D_2, d_{3,m} \geq D_3, d_{4,m} \leq D_4$, and the corresponding positive solutions $\mathbf{u_m}$ to (3.1) such that $\max_{\bar{\Omega}} u_{2m} \to 0$, where $\mathbf{u_m} = (u_{1m}, u_{2m}, u_{3m})^T$ satisfies

$$\begin{cases} -d_{1,m} \Delta u_{1m} = u_{1m}(1 - u_{1m} - \frac{u_{2m}}{u_{1m}+l}), & x \in \Omega, \\ -d_{2,m} \Delta u_{2m} = u_{2m}(-b + \frac{au_{1m}}{u_{1m}+l} - eu_{2m} - \frac{u_{3m}}{u_{2m}+m}), & x \in \Omega, \\ -d_{3,m} \Delta(u_{3m} + d_{4,m} u_{2m} u_{3m}) = u_{3m}(c - \frac{u_{3m}}{u_{2m}+n}), & x \in \Omega, \\ \partial_n u_{1m} = \partial_n u_{2m} = \partial_n u_{3m} = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

Since that $\max_{\bar{\Omega}} u_i > 0$ for $i = 1, 3$, we may assume that $\max_{\bar{\Omega}} u_{1m} \to \overline{u}_1$ and $\max_{\bar{\Omega}} u_{3m} \to \overline{u}_3$. where $\overline{u}_1$ and $\overline{u}_3$ are positive constants. We also claim that u_{1m} and u_{3m} converge uniformly to positive constants respectively.

In fact, there are two cases of $\{d_{3,m}\}_{m=1}^{\infty}$ to be considered.

Case (i) $\{d_{3,m}\}_{m=1}^{\infty}$ is bounded with respect to m. Set

$$\psi_m = d_{3,m}(u_{3m} + d_{4,m} u_{2m} u_{3m}).$$

By the uniform upper bounded of $\mathbf{u_m}$, it is easy to see that $\|\psi_m\| \le C$ for all $m \ge 1$. Since ψ_m satisfies

$$\begin{cases} \Delta\psi_m + \frac{cu_{2m} - u_{3m} - cn}{d_{3,m}(1+d_{4,m}u_{2m})(u_{2m}+n)}\psi_m = 0, & x \in \Omega, \\ \partial_n \psi_m = 0, & x \in \partial\Omega, \end{cases}$$

by L^p estimate and the Sobolev embedding theorems we have

$$\|\psi_m\|_{C^{1,\alpha}(\bar{\Omega})} \le C\|\psi_m\|_{W_{2,p}(\bar{\Omega})} \le C.$$

Similarly, the $C^{2,\alpha}(\bar{\Omega})$ norms of ψ_m is uniformly bounded with respect to m. Thus by passing to subsequence if necessary we may assume that $\psi_m \to \psi$ in $C^2(\bar{\Omega})$. By the definition of ψ_m, for sufficiently large m we have

$$u_{3m} = \frac{\psi_m}{d_{3,m}(1 + d_{4,m}u_{2m})}.$$

Since $u_{2m} \to 0$ and $d_{4,m} \le D_4$, we have

$$u_{3m} \to u_3 \equiv \frac{\psi}{d_3}.$$

Hence ψ satisfies

$$\begin{cases} d_3^2\Delta\psi + \psi(cd_3 - \frac{\psi}{n}) = 0, & x \in \Omega, \\ \partial_n\psi = 0, & x \in \partial\Omega. \end{cases}$$

Hence, $\psi \equiv cnd_3$ otherwise $\psi \equiv 0$ on $\bar{\Omega}$, that implies that $u_3 = \overline{u}_3 = 0$, which is a contradiction to $\overline{u}_3 > 0$, and therefore $u_3 \equiv \psi/d_3 = cn$.

Case (ii) $\{d_{3,m}\}_{m=1}^{\infty}$ is unbounded with respect to m. We may assume that $d_{3,m} \to \infty$. Set

$$\phi_m = u_{3m} + d_{4,m}u_{2m}u_{3m}.$$

Then arguing similarly as above, we obtain

$$-\Delta\phi = 0 \ \texttt{in} \ \Omega, \qquad \partial_n\phi = 0 \ \texttt{on} \ \partial\Omega.$$

Hence, $\phi = u_3 \equiv \texttt{constant} > 0$.

Similarly, we can prove that $u_1 \equiv \texttt{constant} \ge 0$. The above argument shows that there exist positive constants $c_1, c_3 \ge 0$, such that $(u_{1m}, u_{2m}, u_{3m}) \to (c_1, 0, c_3)$. This is a contradiction to Lemma 3.1, thus the proof is completed. □

3.2 Existence of nonhomogeneous positive steady states

In this subsection, we shall discuss the existence of nonhomogeneous positive solutions. Let $\mathbf{X}$ be defined as in Notation 2.1. Define

$$\begin{aligned}\mathbf{X}^+ &= \{\mathbf{u} \in \mathbf{X} : \ u_i > 0 \ \text{ on } \ \bar{\Omega}, \ i = 1, 2, 3\},\\ B(C) &= \{\mathbf{u} \in \mathbf{X} : \ C^{-1} < u_i < C \ \text{ on } \ \bar{\Omega}, \ i = 1, 2, 3\}, C > 0.\end{aligned}$$

Let $\Phi(u) = (d_1u_1, d_2u_2, d_3u_3(1 + d_4u_2))^T$. Then (3.1) is transformed into the form

$$\begin{cases} -\Delta\Phi(\mathbf{u}) = \mathbf{G}(\mathbf{u}), & x \in \Omega,\\ \partial_n \mathbf{u} = 0, & x \in \partial\Omega. \end{cases} \tag{3.8}$$

Since the determinant of $\Phi_u(\mathbf{u})$ is positive for all non-negative $\mathbf{u}$, $\Phi_u^{-1}(\mathbf{u})$ exists and $det\Phi_u^{-1}(\mathbf{u})$ is positive. Hence, $\mathbf{u}$ is a positive solution to (3.8) if and only if

$$F(\mathbf{u}) \triangleq \mathbf{u} - (I - \Delta)^{-1}\{\Phi_u^{-1}[\mathbf{G}(\mathbf{u}) + \nabla\mathbf{u} \cdot \Phi_{uu}(\mathbf{u}) \cdot \nabla\mathbf{u}] + \mathbf{u}\} = 0 \ \text{ in } \ \mathbf{X}^+,$$

where $(I - \Delta)^{-1}$ is the inverse of $I - \Delta$ in $\mathbf{X}$. As $F(\cdot)$ is a compact perturbation of the identity operator, for any $B = B(C)$, the Leray-Schauder degree $deg(F(\cdot), 0, B)$ is well-defined if $F(\mathbf{u}) \neq 0$ on ∂B.

Furthermore, we note that

$$D_u F(\tilde{\mathbf{u}}) = I - (I - \Delta)^{-1}\{\Phi_u^{-1}(\tilde{\mathbf{u}})\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}}) + I\},$$

and recall that if $D_u F(\tilde{\mathbf{u}})$ is invertible, by Theorem 2.8.1 of [39], the index of F at $\tilde{\mathbf{u}}$ is defined a $index(F(\cdot), \tilde{\mathbf{u}}) = (-1)^\gamma$, where γ is the number of negative eigenvalues of $D_u F(\tilde{\mathbf{u}})$.

We refer to the decomposition in our discussion of the eigenvalues of $D_u F(\tilde{\mathbf{u}})$. First, we note that, for each integer $i \geq 1$ and each integer $1 \leq j \leq dim E(\mu_i)$, $\mathbf{X_{ij}}$ is invariant under $D_u F(\tilde{\mathbf{u}})$, and λ is an eigenvalue of $D_u F(\tilde{\mathbf{u}})$ on $\mathbf{X_{ij}}$ if and only if it is an eigenvalue of the matrix

$$I - \frac{1}{1 + \mu_i}[\Phi_u^{-1}\tilde{\mathbf{u}}\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}}) + I] = \frac{1}{1 + \mu_i}[\mu_i I - \Phi_u^{-1}(\tilde{\mathbf{u}})\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}})].$$

Thus, $D_u F(\tilde{\mathbf{u}})$ is invertible if and only if, for all $i \geq 1$, the matrix $I - \frac{1}{1+\mu_i}[\Phi_u^{-1}\tilde{\mathbf{u}}\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}}) + I]$ is non-singular. Writing

$$H(\mu) = H(\tilde{\mathbf{u}}; \mu) \triangleq det\{\mu I - \Phi_u^{-1}(\tilde{\mathbf{u}})\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}})\}, \tag{3.9}$$

we note, furthermore, that if $H(\mu_i) \neq 0$, then for each $1 \leq j \leq dim E(\mu_i)$, the number of negative eigenvalue of $D_u F(\tilde{\mathbf{u}})$ on $\mathbf{X_{ij}}$ is odd if and only if $H(\mu_i) < 0$. In conclusion, we have the following results.

Proposition 3.3 *Suppose that, for all $i \geq 1$, the matrix $\mu_i I - \Phi_u^{-1}(\tilde{\mathbf{u}}) G_u(\tilde{\mathbf{u}})$ is non-singular. Then*

$$index(F(\cdot), \tilde{\mathbf{u}}) = (-1)^{\tau}, \quad where \ \tau = \sum_{i \geq 1, H(\mu_i) < 0} dim E(\mu_i).$$

To facilitate our computation of $index(F(\cdot), \tilde{\mathbf{u}})$, we shall consider carefully the sign of $H(\mu_i)$. In particular, as the purpose of this section is to study the existence of positive solution of (3.8) with respect to the cross-diffusion coefficient d_2, we shall concentrate on the dependence of $H(\mu_i)$ on d_2, and consider d_1, d_3 and d_4 are fixed. At this point, denote that

$$H(\mu) = det\{\Phi_u^{-1}(\tilde{\mathbf{u}})\} det\{\mu \Phi_u \tilde{\mathbf{u}} - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\}.$$

Since we have already established that $det\{\Phi_u^{-1}(\tilde{\mathbf{u}})\}$ is positive, we only need to consider $det\{\mu \Phi_u \tilde{\mathbf{u}} - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\}$. In fact, the value of $\overline{B}_3$ which is given in (2.6) is equal to $det\{\mu \Phi_u \tilde{\mathbf{u}} - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\}$. In the proof of Theorem 2.3, we have discussed the sufficient conditions such that $\overline{B}_3 < 0$. Therefore we have the following proposition.

Proposition 3.4 *Assume that (H_3) holds. Then there exists a positive number d_3^* such that, for all $d_3 \geq d_3^*$, the three roots $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{\mu}_3$ of $det\{\mu \Phi_u \tilde{\mathbf{u}} - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\} = 0$ are all real and satisfy (2.8). Moreover, for all $d_3 \geq d_3^*$*

$$\begin{cases} -\infty < \tilde{\mu}_1 < 0 < \tilde{\mu}_2 < \tilde{\mu}_3, \\ det\{\mu \Phi_u \tilde{\mathbf{u}} - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\} < 0, \quad when \ \ \mu \in (-\infty, \tilde{\mu}_1) \cup (\tilde{\mu}_2, \tilde{\mu}_3), \\ det\{\mu \Phi_u \tilde{\mathbf{u}} - \mathbf{G}_{\mathbf{u}}(\tilde{\mathbf{u}})\} > 0, \quad when \ \ \mu \in (\tilde{\mu}_1, \tilde{\mu}_2) \cup (\tilde{\mu}_3, \infty). \end{cases} \tag{3.10}$$

In order to discuss the effect of cross-diffusion on the existence of nonhomogeneous positive solution to (3.8), we first give a non-existence result when the cross-diffusion term is absent.

Theorem 3.3 *Suppose that*

$$d_1 \geq \frac{1}{\mu_2}, \quad d_3 \geq \frac{c}{\mu_2}, \tag{3.11}$$

where μ_2 is given in Notation 2.1. Then there exists some positive constant $\overline{D}_2$. When $d_2 \geq \overline{D}_2$, without cross-diffusion d_4, (3.1) has no non-constant positive solution. Furthermore

$$u_i \equiv \overline{u}_i, \quad i = 1, 2, 3 \ \ where \ \overline{u}_i \triangleq \frac{1}{measure(\Omega)} \int_{\Omega} u_i.$$

Proof Assume that $\mathbf{u} = (u_1, u_2, u_3)$ is a positive solution of (3.1) with $d_4 = 0$. Let $\overline{u}_i = \frac{1}{measure(\Omega)} \int_{\Omega} u_i$ for $i = 1, 2, 3$. Multiplying the ith equation of (3.1) by $u_i - \overline{u}_i$, and integrating the results over Ω by parts, we have

$$
\begin{aligned}
d_1 \int_\Omega |\nabla u_1|^2 &= \int_\Omega (u_1 - \overline{u}_1)(u_1 g_1(u_1, u_2) - \overline{u}_1 g_1(\overline{u}_1, \overline{u}_2)) \\
&= \int_\Omega (u_1 - \overline{u}_1)\left[(u_1 - \overline{u}_1) - (u_1 + \overline{u}_1)(u_1 - \overline{u}_1)\right. \\
&\quad \left. - \frac{u_1 u_2(\overline{u}_1 + l) - \overline{u}_1 \overline{u}_2 (u_1 + l)}{(u_1 + l)(\overline{u}_1 + l)}\right] \\
&= \int_\Omega (u_1 - \overline{u}_1)^2 \left(1 - u_1 - \overline{u}_1 - \frac{l u_2}{(u_1 + l)(\overline{u}_1 + l)}\right) \\
&\quad - \int_\Omega (u_1 - \overline{u}_1)(u_2 - \overline{u}_2)(l\overline{u}_1 + u_1 \overline{u}_1).
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
d_2 \int_\Omega |\nabla u_2|^2 &= \int_\Omega (u_2 - \overline{u}_2)(u_2 g_2(u_1, u_2, u_3) - \overline{u}_2 g_2(\overline{u}_1, \overline{u}_2, \overline{u}_3)) \\
&= \int_\Omega (u_2 - \overline{u}_2)^2 \left(-b - e u_2 - e\overline{u}_2 + \frac{a u_1 \overline{u}_1 + a l u_1}{(u_1 + l)(\overline{u}_1 + l)} - \frac{m u_3}{(u_2 + m)(\overline{u}_2 + m)}\right) \\
&\quad + \int_\Omega \frac{a l \overline{u}_2 (u_1 - \overline{u}_1)(u_2 - \overline{u}_2)}{(u_1 + l)(\overline{u}_1 + l)} - \int_\Omega \frac{(m\overline{u}_2 + u_2 \overline{u}_2)(u_2 - \overline{u}_2)(u_3 - \overline{u}_3)}{(u_2 + m)(\overline{u}_2 + m)}.
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
d_3 \int_\Omega |\nabla u_3|^2 &= \int_\Omega (u_3 - \overline{u}_3)(u_3 g_3(u_2, u_3) - \overline{u}_3 g_3(\overline{u}_2, \overline{u}_3)) \\
&= \int_\Omega (u_3 - \overline{u}_3)^2 (c - \frac{(\overline{u}_2 + n)(u_3 + \overline{u}_3)}{(u_2 + n)(\overline{u}_2 + n)}) + \int_\Omega \frac{\overline{u}_3^2 (u_2 - \overline{u}_2)(u_3 - \overline{u}_3)}{(u_2 + n)(\overline{u}_2 + n)}.
\end{aligned}
\tag{3.14}
$$

Then we have

$$
\begin{aligned}
& d_1 \int_\Omega |\nabla u_1|^2 + d_2 \int_\Omega |\nabla u_2|^2 + d_3 \int_\Omega |\nabla u_3|^2 \\
&= \int_\Omega (u_1 - \overline{u}_1)^2 \left(1 - u_1 - \overline{u}_1 - \frac{l u_2}{(u_1 + l)(\overline{u}_1 + l)}\right) \\
&\quad + \int_\Omega (u_2 - \overline{u}_2)^2 \left(-b - e u_2 - e\overline{u}_2 + \frac{a u_1 \overline{u}_1 + a l u_1}{(u_1 + l)(\overline{u}_1 + l)} - \frac{m u_3}{(u_2 + m)(\overline{u}_2 + m)}\right) \\
&\quad + \int_\Omega (u_3 - \overline{u}_3)^2 \left(c - \frac{(\overline{u}_2 + n)(u_3 + \overline{u}_3)}{(u_2 + n)(\overline{u}_2 + n)}\right)
\end{aligned}
$$

$$+\int_\Omega (u_1-\overline{u}_1)(u_2-\overline{u}_2)\left[(l\overline{u}_1+u_1\overline{u}_1)+\frac{al\overline{u}_2}{(u_1+l)(\overline{u}_1+l)}\right]$$
$$+\int_\Omega (u_2-\overline{u}_2)(u_3-\overline{u}_3)\left[-\frac{u_2\overline{u}_2+m\overline{u}_2}{(u_2+m)(\overline{u}_2+m)}+\frac{\overline{u}_3^2}{(u_2+n)(\overline{u}_2+n)}\right]. \quad (3.15)$$

By Cauchy inequality, we see that

$$\begin{aligned}
& d_1\int_\Omega |\nabla u_1|^2+d_2\int_\Omega |\nabla u_2|^2+d_3\int_\Omega |\nabla u_3|^2\\
&\le \int_\Omega (u_1-\overline{u}_1)^2\left(1-u_1-\overline{u}_1-\frac{lu_2}{(u_1+l)(\overline{u}_1+l)}+\varepsilon\right)\\
&\quad+\int_\Omega (u_3-\overline{u}_3)^2\left(c-\frac{(\overline{u}_2+n)(u_3+\overline{u}_3)}{(u_2+n)(\overline{u}_2+n)}+\varepsilon\right)\\
&\quad+\int_\Omega (u_2-\overline{u}_2)^2\left(-b-eu_2-e\overline{u}_2+\frac{au_1\overline{u}_1+alu_1}{(u_1+l)(\overline{u}_1+l)}-\frac{mu_3}{(u_2+m)(\overline{u}_2+m)}\right)\\
&\quad+\frac{1}{4\varepsilon}\int_\Omega (u_2-\overline{u}_2)^2\Bigg((u_1-\overline{u}_1)^2\left[(l\overline{u}_1+u_1\overline{u}_1)+\frac{al\overline{u}_2}{(u_1+l)(\overline{u}_1+l)}\right]^2\\
&\quad+(u_3-\overline{u}_3)^2\left[-\frac{u_2\overline{u}_2+m\overline{u}_2}{(u_2+m)(\overline{u}_2+m)}+\frac{\overline{u}_3^2}{(u_2+n)(\overline{u}_2+n)}\right]^2\Bigg). \qquad (3.16)
\end{aligned}$$

where ε is an arbitrary positive constant.

In view of poincaré Inequality we have

$$\begin{aligned}
& d_1\int_\Omega |\nabla u_1|^2+d_2\int_\Omega |\nabla u_2|^2+d_3\int_\Omega |\nabla u_3|^2\\
&\ge \int_\Omega d_1\mu_2(u_1-\overline{u}_1)^2+\int_\Omega d_2\mu_2(u_2-\overline{u}_2)^2+\int_\Omega d_3\mu_2(u_3-\overline{u}_3)^2. \qquad (3.17)
\end{aligned}$$

Since that the upper bound (3.2), we can choose a sufficiently small ε_0 such that

$$\begin{aligned}
d_1\mu_2 &> 1-u_1-\overline{u}_1-\frac{lu_2}{(u_1+l)(\overline{u}_1+l)}+\varepsilon,\\
d_3\mu_2 &> \left(c-\frac{(\overline{u}_2+n)(u_3+\overline{u}_3)}{(u_2+n)(\overline{u}_2+n)}+\varepsilon\right).
\end{aligned}$$

Lastly, by taking $\overline{D}_2 > \frac{1}{\mu_2}((-b-eu_2-e\overline{u}_2+\frac{au_1\overline{u}_1+alu_1}{(u_1+l)(\overline{u}_1+l)}-\frac{mu_3}{(u_2+m)(\overline{u}_2+m)})+\frac{1}{4\varepsilon_0}(u_1-\overline{u}_1)^2[(l\overline{u}_1+u_1\overline{u}_1)+\frac{al\overline{u}_2}{(u_1+l)(\overline{u}_1+l)}]^2+\frac{1}{4\varepsilon_0}(u_3-\overline{u}_3)^2[-\frac{u_2\overline{u}_2+m\overline{u}_2}{(u_2+m)(\overline{u}_2+m)}+\frac{\overline{u}_3^2}{(u_2+n)(\overline{u}_2+n)}]^2)$, we can conclude that $u_i=\overline{u}_i$ for $i=1,2,3$, which finishes the proof. □

From Theorem 3.3 we know that, when cross-diffusion $d_3d_4\Delta(u_2u_3)$ is absent, (3.1) has no non-constant positive solution, whereas Theorem 3.4 shows that the presence of cross-diffusion creates nonhomogeneous solution. In the following text we shall discuss the global existence of the nonhomogeneous solution to (3.8) with respect to d_3 as the other parameters d_1, d_2, d_4 are fixed. Our results are as follows:

Theorem 3.4 *Let d_1, d_2, d_4 be fixed and satisfy that (H_2) and (H_3). Let $\tilde{\mu}$ be defined in (2.8). If $\tilde{\mu}\in(\mu_n,\mu_{n+1})$ for some $n\geq 1$, and the sum $\sigma_n=\sum_{i=2}^{n} dim E(\mu_i)$ is odd. Then there exists a positive number d_3^* such that, if $d_3\geq d_3^*$, (1.1) has at least one nonhomogeneous positive steady state solution.*

Proof Let D_1, D_2, D_3 and D_4 be positive constants and satisfy that $D_1<d_1, D_2<d_2$, $D_3<d_3$ and $D_4>d_4$. By (2.8) and Proposition 3.4, there exists a positive constant d_3^* such that, when $d_3\geq d_3^*$, (3.10) holds and

$$0=\mu_1<\tilde{\mu}_2<\mu_2,\quad \tilde{\mu}_3\in(\mu_n,\mu_{n+1}). \tag{3.18}$$

We shall prove that for $d_3\geq d_3^*$, (3.1) has at least one nonhomogeneous positive solution. The proof, which is by contradiction, is based on the homotopy invariance of the topological degree. Suppose on the contrary that the assertion is not true.

For $t\in[0,1]$, define

$$\begin{aligned}\Phi(t;\mathbf{u})=(&\hat{d}_1u_1+t(d_1-\hat{d}_1)u_1,\hat{d}_2u_2+t(d_2-\hat{d}_2)u_2,\\ &\times(\hat{d}_3+t(\hat{d}_3-d_3))(u_3+td_4u_2u_3))^T,\end{aligned}$$

where $\hat{d}_2=\overline{D}_2$ (which is defined in Theorem 3.3), $\hat{d}_1=\frac{1}{\mu_2}$, $\hat{d}_3=\frac{c}{\mu_2}$, and consider the problem

$$\begin{cases}-\Delta\Phi(t;\mathbf{u})=\mathbf{G}(\mathbf{u}) & \mathtt{in}\ \ \Omega,\quad 0\leq t\leq 1,\\ \partial_n\mathbf{u}=0, & \mathtt{on}\ \ \partial\Omega.\end{cases} \tag{3.19}$$

Then $\mathbf{u}$ is a positive nonhomogeneous solution of (3.1) if and only if it is such a solution of (3.19) for $t=1$. It is clear that $\tilde{\mathbf{u}}$ is the unique constant positive solution of (3.19) for any $0\leq t\leq 1$. As we observed, for any $0\leq t\leq 1$, $\mathbf{u}$ is a positive solution of (3.19) if and only if

$$F(t;\mathbf{u})\triangleq\mathbf{u}-(\mathbf{I}-\Delta)^{-1}\{\Phi_u^{-1}[\mathbf{G}(\mathbf{u})+\nabla\mathbf{u}\cdot\Phi_{uu}(t;\mathbf{u})\cdot\nabla\mathbf{u}]+\mathbf{u}\}=0\ \ \mathtt{in}\ \ \mathbf{X}^+.$$

It is obvious that $F(1;\mathbf{u})=F(\mathbf{u})$. Theorem 3.3 shows that $\mathbf{F}(0;\mathbf{u})=0$ has only the positive solution $\tilde{\mathbf{u}}$ in $\mathbf{X}^+$. By a direct computation,

$$\begin{aligned}D_uF(0;\tilde{\mathbf{u}})&=I-(I-\Delta)^{-1}\{\mathcal{D}^{-1}\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}})+I\}\\ D_uF(1;\tilde{\mathbf{u}})&=I-(I-\Delta)^{-1}\{\Phi_u^{-1}(\tilde{\mathbf{u}})\mathbf{G}_\mathbf{u}(\tilde{\mathbf{u}})+I\}=D_uF(\tilde{\mathbf{u}}).\end{aligned}$$

where $\mathcal{D} = diag(\overline{d}_1, \hat{d}_2, \hat{d}_3)$. In view of Proposition 3.4 and (3.18) , it follows that

$$\begin{cases} H(\mu_1) = H(0) > 0, & \\ H(\mu_i) < 0, & 2 \leq i \leq n, \\ H(\mu_i) > 0, & i \geq n+1. \end{cases}$$

Therefore, zero is not a eigenvalue of the matrix $\mu_i I - \Phi_u^{-1}(\tilde{\mathbf{u}})\mathbf{G_u}(\tilde{\mathbf{u}})$ for all $i \geq 1$, and

$$\sum_{i\geq 1, H(\mu_i)<0} dimE(\mu_i) = \sum_{i=2}^{n} dimE(\mu_i) = \sigma_n, \quad \texttt{which is odd.}$$

Thanks to Proposition 3.3, we have

$$index(F(1;\cdot), \tilde{\mathbf{u}}) = (-1)^{\tau} = (-1)^{\sigma_n} = -1. \tag{3.20}$$

Similarly we can get that

$$index(F(0;\cdot), \tilde{\mathbf{u}}) = (-1)^0 = 1. \tag{3.21}$$

Now by Theorems 3.1, 3.2, there exists a positive constant C such that, for all $0 \leq t \leq 1$, the positive solutions of (3.19) satisfying $\frac{1}{C} \leq u_1, u_2, u_3 \leq C$. Therefore, $F(t; \mathbf{u}) \neq 0$ on ∂B for all $0 \leq t \leq 1$. By the homotopy invariance of the topological degree,

$$deg(F(1;\cdot), 0, B(C)) = deg(F(0;\cdot), 0, B(C)). \tag{3.22}$$

On the other hand, by our supposition, both equations $F(1; \mathbf{u}) = 0$ and $F(0; \mathbf{u}) = 0$ have only the positive solution $\tilde{\mathbf{u}}$ in $B(C)$, and hence, by (3.20) and (3.21),

$$\begin{aligned} deg(F(0;\cdot), 0, B(C)) &= index(F(0;\cdot), \tilde{\mathbf{u}}) = 1, \\ deg(F(1;\cdot), 0, B(C)) &= index(F(1;\cdot), \tilde{\mathbf{u}}) = -1. \end{aligned}$$

which contradicts (3.22) and the proof is complete. □

4 Concluding remarks

The main aim of this paper is to study the spatial pattern formation of (1.1). We show that under some conditions the model generates spatial patterns only in the presence of cross-diffusion. This phenomenon can be regarded as the extension of Turing patterns. Moreover, numerical simulation is given to illustrate the detailed structure of the spatial patterns. Figures 2 and 3 show that the spatial patterns under the special parameters in the text are the same wavenmber striped patterns while the scale of the domain is different. Moreover, when the other parameters are chosen in the numerical simulations, the model may have the different type patterns, such as the spotted patterns.

Note that spatial pattern formation is the process where large-scale ordered spatial patterns emerge from disordered initial conditions. In mathematical sense, ordered spatial patterns are nonuniform steady state solutions. Hence, we also prove that large cross-diffusion induces at least a nonhomogeneous stationary solution. Our systematic research mathematically explains why ecosystem exists the segregation phenomena.

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